Preface

A wide range of engineering applications uses triangulations or meshes as spatial support.

Given a set of points in $\mathbb{R}^d$ ($d \geq 2$), a triangulation of this cloud of points fills the corresponding convex hull with a set of elements which are, in general, simplicial in nature (triangle for $d = 2$ and tetrahedron for $d = 3$), such that some properties are satisfied. Conversely, given a polygonal domain ($d = 2$) or a polyhedral domain ($d = 3$), a mesh of this domain covers the domain with simple geometric elements (triangle, quadrilateral, in two dimensions and tetrahedron, pentahedron, hexahedron, in three dimensions) such that some adequate properties hold.

Numerous computational geometry papers and books are devoted to the algorithms used to construct triangulations, with special attention paid to those resulting in Delaunay triangulations. These algorithms, in a suitable way, are an important part of the algorithms used to construct meshes. To this end, triangulation algorithms are of great interest for defining meshing algorithms.

A large portion of scientific computing in engineering is the solution of partial differential equations of various type (for solid mechanics, fluid mechanics, thermal modeling, ...) by means of the finite element method. This method requires a mesh of the domain upon which the equations are formulated. Thus, meshing algorithms are of major importance in every numerical simulation based on the finite element method. In particular, the accuracy and even the validity of a solution is strongly tied to the properties of the underlying mesh of the domain under consideration.

The aim of this book is to describe, in the first chapters, the different algorithms suitable for constructing a triangulation and, more precisely, a Delaunay triangulation. Then, the following chapters will indicate the way in which triangulation methods can be extended to develop meshing algorithms. Only Delaunay type methods are discussed here while observ-
ing that a large variety of meshing algorithms exists. To this end, the book is divided into three parts. The first part, devoted to triangulations, comprises the first four chapters. The second part dealing with meshing algorithms is made up of the five following chapters and the third part discusses several applications in the four last chapters. A technical appendix and an index are also included in the book.

In Chapter 1, general definitions relative to elements, triangulations and meshes are given. Algorithmic hints are given regarding some key-issues that will be used extensively in the algorithms and methods developed throughout the book.

In Chapter 2, several methods are discussed that result in the construction of a Delaunay triangulation. Given a set of points in $R^d$ with $d = 2$ or $d = 3$, we propose several methods that make the construction of the Delaunay triangulation possible. The definition of a Delaunay triangulation is first given and then several construction methods are presented. A "popular" method, referred to as the incremental method, is emphasized and a reduced version of it is discussed in detail. Other approaches are also given. Algorithmic or computational aspects of the reduced method are mentioned, while indicating the numerical difficulties that can be expected along with some proposed solutions.

Chapter 3 deals with constrained triangulations. A Delaunay triangulation is given along with a set of constraints. These constraints are indeed a set of edges in two dimensions and a set of edges and faces in three dimensions. The question is then how to enforce these entities into the triangulation so that they exist, in some sense, as entities of the resulting triangulation. The case of higher dimensions is also mentioned.

Chapter 4 is devoted to the way in which anisotropic triangulations can be obtained. Given a set of points and a specified metric field, the purpose is to construct a triangulation which satisfies the given field. The metric specifies the properties that the triangulation should enjoy, in terms of prescribed sizes and directional information.

The following chapters discuss the way in which arbitrarily shaped domains can be meshed. Algorithms developed to this end are derived from triangulation algorithms as described in the previous chapters. A chapter is devoted to two dimensions, another deals with parametric surfaces, while a third one discusses the three-dimensional case.

The two-dimensional case is detailed in Chapter 5. A domain in $R^2$ is given via a discretization of its boundary, where this boundary is given as a
list of segments. The problem then is how to construct a mesh of the given domain. This construction involves mainly two steps, one being related to the triangulation problem (as described previously), the other dealing with the way in which a suitable set of internal points can be created. The notion of a control space is introduced as a way to govern the creation of the relevant field points, as well as to specify the nature of the expected point to point connections. This framework is discussed for a classical case, where the boundary discretization is the only input data available, for the isotropic case, where desired sizes are specified and, for the pure anisotropic situation, where both directional and size specifications are given.

Chapter 6 indicates how to mesh a parametric surface. A field of metrics is constructed following the fundamental forms of the surface. This field serves to control the mesh construction. In particular, we show how to construct a so-called geometric mesh that is a close approximation of the surface.

Chapter 7 follows the same steps as Chapter 5 while discussing the meshing problem of a domain in $\mathbb{R}^3$. This domain is defined by a discretization of its boundary, in other words, a surface mesh.

Chapter 8 is devoted to mesh optimization, the meshes in question being composed of triangles (in two dimensions) or tetrahedra (in three dimensions). Several local tools are introduced and we propose a strategy that makes the development of a global optimization algorithm possible.

In Chapter 9, mesh adaptivity is discussed by focusing on the computational aspect of this topic. Two approaches are proposed. The first one, which is only discussed briefly, relies on local modifications of an existing mesh. The other approach which is discussed in more depth, relies on the construction of the entire mesh. The general scheme of a fully automatic adaptivity loop is given.

The last chapters show how to use the previous materials and methods in a true application of the finite element method in two dimensions.

Chapter 10 introduces some data structures which enable us to develop a meshing theoretical background. A conceptual data structure is proposed and two applications are discussed. The first one is related to a data structure suitable for describing the geometry of the domain of interest while the second structure corresponds to the mesh representation.

In Chapter 11 some details are given concerning the way in which a $\mathbb{R}^2$ or $\mathbb{R}^3$ domain boundary can be meshed or remeshed. The resulting
mesh serves as input for the meshing methods applied in the corresponding domain.

In Chapter 12, significant mesh examples are given. To this end, several two-dimensional C.F.D. applications are used.

In Chapter 13, several applications of triangulation methods are described which are not necessarily related to the finite element context.

Software packages are listed in the appendix that are mainly devoted to mesh generation or mesh management for finite element purposes. These packages deal with mesh generation for two or three dimensions (most of them are issued from INRIA).

The book ends with an index and a bibliography.

As a conclusion, we would like to thank here those who contributed, in one way or another, to this book. Among these are the members of the Gamma group at INRIA, P.J. Frey, P. Laug, F. Hecht and E. Saltel who have contributed in various ways to this book. We would also like to thank B. Mohammadi and J. Galtier for their help. We are indebted to M. Desnous, P. Joly, A. Marrocco, A. Perronnet and J.D. Boissonnat, whose fruitful comments helped us to improve several aspects of the book.

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